Subject: Statistical Inference   Topic: Bays estimation

Definition:

Let be a $f\left(x/\theta\right) = f(x;\theta)$ given density of a given sample $X_1, X_2, \ldots..X_n$.

Then the posterior density is given by $f\left(\theta/\underline{X}\right) = \dfrac{f\left(\underline{X}/\theta\right)g(\theta)}{\int f\left(\underline{X}/\theta\right)g(\theta)d(\theta)}$

$f\left(\underline{X}/\theta\right) = \prod_{i=1}^{n} f\left(x/\theta\right)$ is the likelihood function and $g(\theta)$ is prior density . Then the bays

$E\left(\mathrm{T}(\theta)/\underline{X}\right) = \int \mathrm{T}(\theta) f\left(\theta/\underline{X}\right) d\theta$ estimatoris given by

Where $\mathrm{T}(\theta)$ is the function of parameter whose

$E\left(\theta/\underline{X}\right) = \int \theta f\left(\theta/\underline{X}\right) d\theta$ baysestimator is to be found. If θ is the parameter then $\mathrm{T}(\theta) = \theta$ whose estimator is to be

computed then (i.e.) $E\left(\theta/\underline{X}\right) = \theta^*$ are required bays

estimators.

Procedure:

If we have probability density functions $f\left(x/\theta\right)$ with parameter $'\theta'$. Then obtain

the likelihood function as $f\left(\underline{X}/\theta\right)$ and multiply it with given prior density. (i.e.) $f\left(\underline{X}/\theta\right)g(\theta)$

where $g(\theta)$ is the prior density then we find $\int f\left(\underline{X}/\theta\right)g(\theta)d\theta$

then obtain the posterior density which is given by $f\left(\theta/\underline{X}\right) = \dfrac{f\left(\underline{X}/\theta\right)g(\theta)}{\int f\left(\underline{X}/\theta\right)g(\theta)d(\theta)}$

Then lastly the bays estimator of $'\theta'$ is given by $\theta^* = E\left(\theta/\underline{X}\right) = \int \theta f\left(\theta/\underline{X}\right) d\theta$

Properties of Bayes Estimation

Following are the properties of bayes estimation

1- Bayes estimator is always a function of minimal sufficient statistic.
2- Bayes estimator is a constant estimator.
3- It is a $"BAN"$ (Best Asymptotic Normal Property) estimator.
4- Bayes estimator is corresponding to the known prior distribution which is optimum.

Loss Function

Let $'T'$ denote on estimator of $"T(\theta)"$ then loss function $\&(t,\theta)$ is defined as real valued function satisfying

(i) $\&(t,\theta) \geq 0$ For all tandθ
(ii) $\&(t,\theta) = 0$ For $t = \mathrm{T}(\theta)$ i.e. it is a measure of error .

Squared Error Loss function

$\&(t,\theta) = [t - T(\theta)]^2$ is called Squared Error Loss function. It increase as the error $t - T(\theta)$ increase in magnitude.

Absolute Error Loss function

$\&(t,\theta) = |t - T(\theta)|$ Is called the absolute error loss function . it is increase as the $[t - T(\theta)]$ increse.

Risk function

For a given loss function $\&(t,\theta)$ the risk function $R_t(\theta)$ of an estimator $t = \&(X_1, X_2, ...X_n.)$ is defined to be $R_t(\theta) = E[\&(t,\theta)]$. i.e. the risk function is the average loss.

Question 1 :

Suppose that the time to failure $'T'$ of an elictric tube in an exponential random variable with parametere $'\theta'$. We also assume the prior density for $'\theta'$ as $g(\theta) = 160e^{-160\theta}$. Then obtain the Bayes estimator.

Solution :-As given the elictric tube in an exponential random variable with parametere $'\theta'$

$$X \approx \exp(\theta)$$

$$f(x;\theta) = f\left(X/_{\theta}\right) = \theta e^{-\theta X} \qquad 0 \le X \ge \infty$$

The likelihood function is

$$f(\underline{X};\theta) = \theta^n e^{-\theta \sum X}$$

And prior density is

$$g(\theta) = 160e^{-160\theta}$$

Now $f(\underline{X};\theta).g(\theta) = 160\theta^n e^{-\theta(160+\sum X)}$

We know that the posterior density function is

$$f\left(\theta/_{\underline{X}}\right) = \frac{f\left(\underline{X}/_{\theta}\right)g(\theta)}{\int_0^{\infty} f\left(\underline{X}/_{\theta}\right)g(\theta)d\theta} \rightarrow (A)$$

$$\int_0^{\infty} f(\underline{X};\theta).g(\theta)d\theta = \int_0^{\infty} 160\theta^n e^{-\theta \sum X} e^{-160\theta} d\theta$$

$$= 160\int_0^{\infty} \theta^n e^{-\theta(160+\sum X)} d\theta$$

$$= 160\int_0^{\infty} \theta^{n+1-1} e^{\frac{-\theta}{(160+\sum X)^{-1}}} d\theta$$

Comparing with gamma function

$$= 160\overline{)n+1}\left[(160+\sum X)^{-1}\right]^{n+1}$$

$$= 160\overline{)n+1}\frac{1}{(160+\sum X)^{n+1}}$$

Now put in eq (A)

$$f\left(\theta/_{\underline{X}}\right) = \frac{160\theta^n e^{-\theta(160+\sum X}}{160\overline{)n+1}\left(\frac{1}{160+\sum X}\right)^{n+1}}$$

$$= \frac{\theta^{n} e^{-\theta(160+\sum X}}{\Gamma(n+1)\left(\frac{1}{160+\sum X}\right)^{n+1}}$$

Now the bayes estimator is

$$E\left({T(\theta)}\big/ {\underline{X}} \right) = \theta^{*} = \int \theta f\left(\frac{\theta}{\underline{X}}\right) d\theta$$

$$= \int_{0}^{\infty} \frac{\theta.\theta^{n} e^{-\theta(160+\sum X)}}{\Gamma(n+1)\left(\frac{1}{160+\sum X}\right)^{n+1}} \, d\theta$$

$$= \frac{1}{\Gamma(n+1)\left(\frac{1}{160+\sum X}\right)^{n+1}} \int_{0}^{\infty} \theta^{n+2-1} e^{\frac{-\theta}{(\sum X+160)^{-1}}} d\theta$$

Comparing with gamma function

$$= \frac{\Gamma(n+2)\left[(\sum X + 160)^{-1}\right]^{n+2}}{\Gamma(n+1)\left(\frac{1}{160+\sum X}\right)^{n+1}}$$

$$= \frac{\Gamma(n+1+1)\left(\frac{1}{160+\sum X}\right)^{n+2}}{\Gamma(n+1)\left(\frac{1}{160+\sum X}\right)^{n+1}}$$

$$= \frac{(n+1)\Gamma(n+1)}{\Gamma(n+1)}\left(\frac{1}{160+\sum X}\right)^{n+1}$$

$$E\left({T(\theta)}\big/ {\underline{X}} \right) = \frac{n+1}{160+\sum X} \qquad \therefore n+2-n-1$$

$$\therefore 1$$

$$= \frac{n+1}{160+n\overline{\overline{X}}}$$

So $\theta^{*} = \frac{n+1}{160+n\overline{\overline{X}}}$ is the bayes estimator of the given parameter $'\theta'$. .

Question 2 :

Assume that the number of time $'X'$ that the weight switch can be turned on population until fails in geometric random variable with parameter $'P'$. The prior density of unknown parameter $'P'$ is $g(p) = \frac{\rceil a+b}{\rceil a \rceil b} P^{a-1}(1-P)^{b-1}; 0 \le P \ge 1.$

Here $a = 3.995, b = 3991, \sum X_i = 1000$

For a random sample of size "25". Find P^* for P?

In geometric distribution

$$f\left(X/P\right) = pq^{x-1}$$

Taking likelihood function

$$f\left(\frac{X}{P}\right) = p^n q^{\sum (X-1)}$$

$$f\left(\frac{X}{P}\right) = p^n q^{\sum X-n}$$

The prior density is

$$g(p) = \frac{\rceil a+b}{\rceil a \rceil b} P^{a-1}(1-P)^{b-1}$$

Now $$f\left(\frac{X}{P}\right) g(p) = \frac{\rceil a+b}{\rceil a \rceil b} P^{n+a-1}(1-P)^{\sum X-n+b-1}$$

As we that the posterior density

$$f\left(\frac{P}{X}\right) = \frac{f\left(\frac{X}{P}\right) g(P)}{\int_{-\infty}^{\infty} f\left(\frac{X}{P}\right) g(p)\,dP}$$

$$\int_{-\infty}^{\infty} f\left(\frac{X}{P}\right) g(p)\,dp = \frac{\rceil a+b}{\rceil a \rceil b} P^{a+n-1}(1-P)^{b+\sum X-n-1}\,dP$$

Comparing with beta of type 1st

$$=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\beta\left(a+n, b+\sum X-n\right)$$

$$=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+n)\Gamma\left(b+\sum X-n\right)}{\Gamma\left(a+n+b+\sum X-n\right)}$$

$$=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+n)\Gamma\left(b+\sum X-n\right)}{\Gamma\left(a+b+\sum X\right)}$$

Put in Esq. (A)

$$f\left(\theta/\underline{X}\right)=\frac{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}P^{n+a-1}(1-p)^{\sum X-n+b-1}}{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+n)\Gamma\left(b+\sum X-n\right)}{\Gamma\left(a+b+\sum X\right)}}$$

$$=\frac{\Gamma\left(a+b+\sum X\right)P^{n+a-1}(1-p)^{\sum X-n+b-1}}{\Gamma(n+a)\Gamma\left(b+\sum X-n\right)}$$

Now the bayes estimator is

$$P^{*}=E\left(T(P)/\underline{X}\right)=\int f\left(P/\underline{X}\right)dP$$

$$=\frac{\Gamma\left(a+b+\sum X\right)}{\Gamma(n+a)\Gamma\left(b+\sum X-n\right)}\int_{0}^{1}P^{a+n-1-1}(1-p)^{\sum X-n+b-1}dP$$

Comparing with beta type 1st

$$P^{*}=\frac{\Gamma\left(a+b+\sum X\right)}{\Gamma(n+a)\Gamma\left(b+\sum X-n\right)}\frac{\Gamma(a+n+1)\Gamma\left(\sum X-n+b\right)}{\Gamma\left(a+n+1+\sum X-n+b\right)}$$

$$=\frac{\Gamma\left(a+b+\sum X\right)}{\Gamma(n+a)\Gamma\left(\sum X+b-n\right)}\frac{(a+n)\Gamma(a+n)\Gamma\left(\sum X+b-n\right)}{\left(\sum X+a+b\right)\Gamma\left(\sum X+a+b\right)}$$

$$P^{*}=\frac{(a+n)}{\left(\sum X+a+b\right)}$$

Which is required bayes estimator of P

Now substitute the given values for P^*

$n = 25, a = 3.995, b = 3991, \sum X_i = 1000$

$$P^* = \frac{3.995 + 25}{1000 + 3.991 + 3.995}$$

$$= \frac{28.995}{4994.995}$$

$$= 0..005805$$

Which is the required result .

<u>Question 3 :</u>

Let $X_1, X_2, ...X_n$ be a random sample from a bernulli distribution with p.d.f $f\left(X/\theta\right) = \theta^X (1-\theta)^{1-X}$. Assumethat the prior density of θ is given by $g(\theta) = \mathrm{I}_{(0,1)}(\theta)$ (i.e) θ is uniformly distributed over the interval (0,1). Obtain the posterior bayes estimator of θ and $\mathrm{T}(\theta) = \theta(1-\theta)$ and also that the estimators are not unbiased?

The p.d.f of bernulli distribution is

$$f\left(X/\theta\right) = \theta^X (1-\theta)^{1-X}$$

Then L.H.F is

$$f\left(\underline{X}/\theta\right) = \theta^{\sum X} (1-\theta)^{n-\sum X}$$

And prior density function

$$g(\theta) = 1$$

$$f\left(\underline{X}/\theta\right) g(\theta) = \theta^{\sum X} (1-\theta)^{n-\sum X}$$

As the posterior density function is

$$f\left(\theta/\underline{X}\right) = \frac{f\left(\underline{X}/\theta\right) g(\theta)}{\int f\left(\underline{X}/\theta\right) g(\theta) d(\theta)} \longrightarrow (A)$$

Now $\int_{-\infty}^{\infty} f\left(X/P\right) g(\theta)d\theta = \int_0^1 \theta^{\sum X}(1-\theta)^{n-\sum X} d\theta$

$$= \int_0^1 \theta^{\sum X+1-1}(1-\theta)^{n-\sum X+1-1} d\theta$$

Comparing with Beta type first

$$= \beta\left(\sum X+1, n-\sum X+1\right)$$

$$= \frac{\Gamma\left(\sum X+1\right)\Gamma\left(n-\sum X+1\right)}{\Gamma\left(\sum X+1+n-\sum X+1\right)}$$

$$= \frac{\Gamma\left(\sum X+1\right)\Gamma\left(n-\sum X+1\right)}{\Gamma\left(n+2\right)}$$

Put in eq (A)

$$f\left(\theta/\underline{X}\right) = \frac{\theta^{\sum X}(1-\theta)^{n-\sum X}}{\dfrac{\Gamma\left(\sum X+1\right)\Gamma\left(n-\sum X+1\right)}{\Gamma\left(n+2\right)}}$$

$$= \Gamma\left(n+2\right)\frac{\theta^{\sum X}(1-\theta)^{n-\sum X}}{\Gamma\left(\sum X+1\right)\Gamma\left(n-\sum X+1\right)}$$

Now the Bayes Estimator for θ is

$$\theta^* = E\left(\mathrm{T}(\theta)/\underline{X}\right) = \int_0^1 \theta f\left(\theta/\underline{X}\right) d\theta$$

$$= \frac{\Gamma\left(n+2\right)}{\Gamma\left(\sum X+1\right)\Gamma\left(n-\sum X+1\right)}\int_0^1 \theta^{\sum X+2-1}(1-\theta)^{n-\sum X+1-1} d\theta$$

Comparing with Beta type 1st

$$= \frac{\Gamma(n+1+1)}{\Gamma(\sum X+1)\Gamma(n-\sum X+1)} \frac{\Gamma(\sum X+2)\Gamma(n-\sum X+1)}{\Gamma(\sum X+2+n-\sum X+1)}$$

$$= \frac{\Gamma(n+2)}{\Gamma(\sum X+1)} \frac{(\sum X+1)\Gamma(\sum X+1)}{\Gamma(n+3)}$$

$$= \frac{\Gamma(n+2)\left(\sum X+1\right)}{(n+2)\Gamma(n+2)}$$

$\theta^{*} = \dfrac{\sum X+1}{n+2}$ Which is the required bayes estimator for "θ"

Now Bayes estimator for $\theta^{*}(1-\theta)^{*}$ is

$$\theta^{*}(1-\theta)^{*} = E\left(\mathrm{T}(\theta)/\underline{X}\right) = \int_0^1 \theta(1-\theta) f\left(\theta/\underline{X}\right) d\theta$$

$$= \frac{\Gamma(n+2)}{\Gamma(\sum X+1)\Gamma(n-\sum X+1)} \int_0^1 \theta^{\sum X+2-1}(1-\theta)^{n-\sum X+2-1} d\theta$$

Comparing with Beta type 1st

$$= \frac{\Gamma(n+2)}{\Gamma(\sum X+1)\Gamma(n-\sum X+1)} \beta(\sum X+2, n-\sum X+2)$$

$$= \frac{\Gamma(n+2)}{\Gamma(\sum X+1)\Gamma(n-\sum X+1)} \frac{\Gamma(\sum X+2)\Gamma(n-\sum X+2)}{\Gamma(\sum X+2+n-\sum X+2)}$$

$$= \frac{\Gamma(n+2)(\sum X+1)\Gamma(\sum X+1)(n-\sum X+1)\Gamma(n-\sum X+1)}{\Gamma(\sum X+1)\Gamma(n-\sum X+1)\Gamma(n+4)}$$

$$= \frac{\Gamma(n+2)(\sum X+1)(n-\sum X+1)}{(n+3)(n+2)\Gamma(n+2)}$$

$$\theta^*(1-\theta)^* = \frac{(\sum X+1)(n-\sum X+1)}{(n+2)(n+3)}$$ Which is the required Bayes estimator

Now we check that whether estimator is biased or unbiased.

$$\theta^* = \frac{\sum X+1}{n+2} \therefore E(X)=\theta$$

$$E(\theta^*) = \frac{E(\sum X+1)}{n+2}$$

$$= \frac{\sum E(X)+1}{n+2} = \frac{n\theta+1}{2} \neq \theta$$

$$And \theta^*(1-\theta)^* = \frac{(\sum X+1)(\sum X-n+1)}{(n+3)(n+2)}$$

$$E(\theta^*(1-\theta)^*) = \frac{(n\theta+1)(n\theta-n+1)}{(n+3)(n+2)} \neq \theta(1-\theta)$$

So these Bayes estimator are not the unbiased i.e biased estimator

Question: 4

Statement:

If X_i ~pission distribution with parameter $'\theta'$ with assume that a prior density $g(\theta) = \frac{1}{m!}(\frac{m+1}{\theta_o})^{m+1} \theta^m e^{-(m+1)\theta/\theta_o}$ for a random sample of size 'n' .Find the bay's estimator for parameter $'\theta'$.

Solution:- As X $\rightarrow$Pioson distribution with p.d.f

$f\left(m/\theta\right) = \frac{e^{-\theta}\theta^m}{m!}$ m : 1,2,3,…, ∞

As we know that the posterior density is

$$f\left(\theta \big/ \underline{m}\right) = \frac{f\left(\underline{m} \big/ \theta\right) g(\theta)}{\int f\left(\underline{m} \big/ \theta\right) g(\theta) d(\theta)} \longrightarrow \text{(A)}$$

Taking likelihood function of p.d.f

Now $f\left(\underline{m} \big/ \theta\right) = \dfrac{e^{-m\theta}\theta^{\Sigma m}}{\prod_{i=1}^{m} m!}$

The prior density is

$$g(\theta) = \frac{1}{m!}\left(\frac{m+1}{\theta_o}\right)^{m+1} \theta^{m} e^{-(m+1)\theta/\theta_o}$$

$$f\left(\underline{m} \big/ \theta\right) g(\theta) = \frac{e^{-m\theta}\theta^{\Sigma m}}{\prod_{i=1}^{m} m!} \frac{1}{m!}\left(\frac{m+1}{\theta_o}\right)^{m+1} \theta^{m} e^{-(m+1)\theta/\theta_o}$$

$$f\left(\underline{m} \big/ \theta\right) g(\theta) = \frac{\left(\frac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m} m!} \theta^{\Sigma m+m} e^{-\theta\left(m+\frac{m+1}{\theta_o}\right)}$$

$$\int f\left(\underline{m} \big/ \theta\right) g(\theta) = \frac{\left(\frac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m} m!} \int_0^{\infty} \theta^{\Sigma m+m+1-1} e^{-\theta/\left(m+\frac{m+1}{\theta_o}\right)^{-1}} d\theta$$

Comparing with gamma function

$$\int f\left(\underline{m} \big/ \theta\right) g(\theta) = \frac{\left(\frac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m} m!} \overline{\left.\right)\Sigma m + m + 1} \left[\left(m+\frac{m+1}{\theta_o}\right)^{-1}\right]^{\Sigma m+m+1}$$

$$\int f\left(\underline{m}\big/\theta\right)g(\theta)=\frac{\left(\dfrac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m}m!}\overline{)\Sigma m+m+1}\left[\frac{1}{\left(m+\dfrac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+1}$$

Put in (A)

$$f\left(\theta\big/\underline{m}\right)=\frac{\dfrac{\left(\dfrac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m}m!}\theta^{\Sigma m+m}e^{-\theta\left(m+\frac{m+1}{\theta_o}\right)}}{\dfrac{\left(\dfrac{m+1}{\theta_o}\right)^{m+1}}{m!\prod_{i=1}^{m}m!}\overline{)\Sigma m+m+1}\left[\dfrac{1}{\left(m+\dfrac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+1}}$$

$$f\left(\theta\big/\underline{m}\right)=\frac{\theta^{\Sigma m+m}e^{-\theta\left(m+\frac{m+1}{\theta_o}\right)}}{\overline{)\Sigma m+m+1}\left[\dfrac{1}{\left(m+\dfrac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+1}}$$

Now the bayes estimator is

$$\theta^{*}=E\left(\tau(\theta)\big/\underline{m}\right)=\int\theta f\left(\theta\big/\underline{m}\right)d\theta$$

$$=\frac{1}{\overline{)\Sigma m+m+1}\left[\dfrac{1}{\left(m+\dfrac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+1}}\int_{0}^{\infty}\theta^{\Sigma m+m+2-1}e^{-\theta\left(m+\frac{m+1}{\theta_o}\right)}d\theta$$

Comparing with gamma function

$$= \frac{1}{\overline{)\Sigma m+m+1}\left[\frac{1}{\left(m+\frac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+1}} \overline{)\Sigma m+m+2}\left[\frac{1}{\left(m+\frac{m+1}{\theta_o}\right)}\right]^{\Sigma m+m+2}$$

$$= \frac{(\Sigma m+m+1)\overline{)\Sigma m+m+1}}{\overline{)\Sigma m+m+1}}\left(1 \Big/ \left(m+\frac{m+1}{\theta_o}\right)\right)$$

$$\theta^{*} = \frac{(\Sigma m+m+1)}{\left(m+\frac{m+1}{\theta_o}\right)}$$

Which is the required bayes estimator for 'θ'

Question: 5

Statement:

Let $X_1, X_2, X_3, \ldots X_n$ be a random sample from a normal density with mean 'θ' and variance '1' . Consider estimation 'θ' with a square error loss function .Assume that 'θ' has a normal distribution with mean μ_o and variance 1. Obtain Bayes estimator for 'θ' .

As we know that the posterior density is

$$f\left(\theta \big/ \underset{-}{X}\right) = \frac{f\left(\underset{-}{X} \big/ \theta\right) g(\theta)}{\int f\left(\underset{-}{X} \big/ \theta\right) g(\theta) d(\theta)} \longrightarrow \text{(A)}$$

As $\quad X \rightarrow N(\theta,1)$

$$f\left(X/\theta\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}$$

Taking likelihood function

$$f\left(\underline{X}/\theta\right)=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x-\theta)^2}$$

$$f\left(\underline{X}/\theta\right)=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left[\sum x^2+n\theta^2-2\theta\sum x\right]}$$

$$f\left(\underline{X}/\theta\right)=\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\left[\sum x^2+n\theta^2-2\theta n\bar{x}\right]}$$

The prior density is $\theta \to N(\mu_o,1)$

$$g(\theta)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\theta-\mu_o)^2}$$

$$f\left(\underline{X}/\theta\right)g(\theta)=\left(\frac{1}{\sqrt{2\pi}}\right)^{n+1} e^{-\frac{1}{2}\left[\sum x^2+n\theta^2+\theta^2+\mu_o^2=2\theta\mu_o-2\theta n\bar{x}\right]}$$

$$f\left(\underline{X}/\theta\right)g(\theta)=\left(\frac{1}{\sqrt{2\pi}}\right)^{n+1} e^{-\frac{1}{2}\left[\sum x^2+(n+1)\theta^2+\mu_o^2=2\theta(\mu_o+n\bar{x})\right]}$$

$$f\left(\underline{X}/\theta\right)g(\theta)=\left(\frac{1}{\sqrt{2\pi}}\right)^{n+1} e^{-\frac{1}{2}\left[\sum X^2+\mu_o^2\right]}e^{=\frac{1}{2}\left[(n+1)\theta^2-2\theta\left(\mu_o+n\bar{x}\right)\right]}$$

Let $K=\left(\frac{1}{\sqrt{2\pi}}\right)^{n+1} e^{-\frac{1}{2}\left[\sum x^2+\mu_o^2\right]}$

$$f\left(\underline{X}/\theta\right)g(\theta)=K\,e^{=\frac{1}{2}\left[(n+1)\theta^2-2\theta\left(\mu_o+n\bar{x}\right)\right]}$$

$$f\left(\bar{X}/\theta\right)g(\theta)=K\,e^{=\frac{n+1}{2}\left[\theta^2-2\theta\left(\frac{\mu_o+n\bar{x}}{n+1}\right)\right]}$$

Now put in equation (A)

$$f\left(\theta/\bar{x}\right)=\frac{K\,e^{=\frac{n+1}{2}\left[\theta^2-2\theta\left(\frac{\mu_o+n\bar{x}}{n+1}\right)\right]}}{K\int_{-\infty}^{\infty} e^{=\frac{n+1}{2}\left[\theta^2-2\theta\left(\frac{\mu_o+n\bar{x}}{n+1}\right)\right]}d\theta}$$

$$\int_{-\infty}^{\infty} e^{=\frac{n+1}{2}\left[\theta^2-2\theta\left(\frac{\mu_o+n\bar{x}}{n+1}\right)\right]}d\theta=\int_{-\infty}^{\infty} e^{=\frac{n+1}{2}\left[\theta^2-2\theta\left(\frac{\mu_o+n\bar{x}}{n+1}\right)+\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2-\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}d\theta$$

$$=\int_{-\infty}^{\infty} e^{\frac{n+1}{2}\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2}e^{=\frac{n+1}{2}\left[\theta\ -\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}d\theta$$

So,

$$f\left(\theta/\bar{x}\right)=\frac{e^{\frac{n+1}{2}\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2}e^{=\frac{n+1}{2}\left[\theta\ -\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}}{\int_{-\infty}^{\infty} e^{\frac{n+1}{2}\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2}e^{=\frac{n+1}{2}\left[\theta\ -\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}d\theta}$$

$$f\left(\theta/\bar{x}\right)=\frac{e^{=\frac{n+1}{2}\left[\theta\ -\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}}{\int_{-\infty}^{\infty} e^{=\frac{n+1}{2}\left[\theta\ -\left(\frac{\mu_o+n\bar{x}}{n+1}\right)^2\right]}d\theta} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (B)$$

As we know that area under the curve is unity so,

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx=1$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx=\sigma\sqrt{2\pi}$$

Now equation (B) become

$$f\left(\theta \middle/ \underset{-}{x}\right) = \frac{e^{=\frac{1}{2\left(1/_{n+1}\right)}\left[\theta - \left(\frac{\mu_o + n\bar{x}}{n+1}\right)^2\right]}}{\int\limits_{-\infty}^{\infty} e^{=\frac{1}{2\left(1/_{n+1}\right)}\left[\theta - \left(\frac{\mu_o + n\bar{x}}{n+1}\right)^2\right]} d\theta}$$

And

$$\int\limits_{-\infty}^{\infty} e^{=\frac{1}{2\left(1/_{n+1}\right)}\left[\theta - \left(\frac{\mu_o + n\bar{x}}{n+1}\right)^2\right]} d\theta = \sqrt{2\pi}\sqrt{\frac{1}{n+1}}$$

It means that

$$\tau(\theta) = \left(\theta \middle/ \underset{-}{x}\right) \quad \rightarrow N\left(\frac{n\bar{x} + \mu_o}{n+1}, \frac{1}{n+1}\right)$$

Hence $\theta^* = E\left(\theta \middle/ \underset{-}{x}\right) = \frac{n\bar{x} + \mu_o}{n+1}$ is the required Bayes estimator.